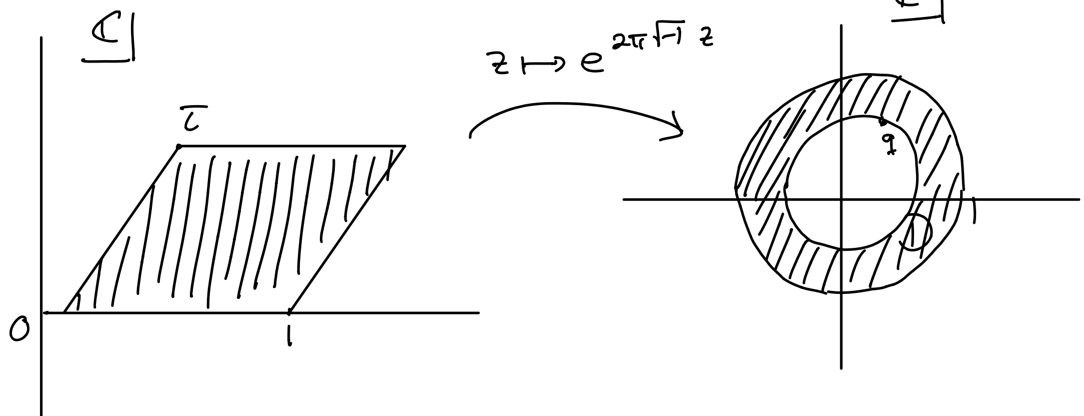
Conformal blocks on torus:

Let τ be an element of the upper half plane

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$$

Denote the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$ by Γ.

Define torus by $E = \mathbb{C}/\Gamma$



Suppose $0 \le \operatorname{Re}\tau < 1$ and set $q = e^{2\pi\sqrt{-1}\,\tau}$

Denote by G the transformation group of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ generated by $f(w) = qw$, $w \in \mathbb{C}^*$. "dilatation"

Then $\mathbb{C}^*/G \cong E$ $\quad (z \mapsto w = e^{2\pi\sqrt{-1}\, z})$

Now set $\zeta = 2\pi\sqrt{-1}\, z$ for the torus $\mathbb{C}^*/G$.

Since $w = e^{\zeta} \longrightarrow S(w,\zeta) = \frac{1}{2}$

Prop. 4 $\longrightarrow T(\zeta) = \sum_{n\in\mathbb{Z}} \left(L_n - \frac{c}{24}\delta_{n,0}\right) e^{-n\zeta}$

$\Rightarrow L_0'$ of torus $\mathbb{C}^*/G$ is given by

$$L_0' = L_0 - \frac{1}{24}c.$$

action of dilatation operator:

$$q^{L_0 - \frac{1}{24}c}\xi = \exp 2\pi\sqrt{-1}\,\tau\left(\Delta_\lambda + d - \frac{c}{24}\right)\xi, \quad \xi \in H_\lambda(d)$$

Take distinct points $p_1, \dots, p_n$ on the torus $\mathbb{C}^*/G$ and represent them as points in D.

$\longrightarrow$ associate level k highest weights $\mu_1, \dots, \mu_n$ to $p_1, \dots, p_n$.

associate H_λ to origin of w-plane and H_λ^* to infinity.

$\longrightarrow$ space of conformal blocks

$$\mathcal{H}(0, p_1, \dots, p_n, \infty; \lambda, \mu_1, \dots, \mu_n, \lambda^*)$$

$\Psi \in \mathcal{H}$ is linear operator

$$\Psi : H_\lambda \otimes H_{\mu_1} \otimes \cdots \otimes H_{\mu_n} \longrightarrow \overline{H_\lambda}.$$

Consider

$$\mathrm{Tr}_{H_\lambda} q^{L_0 - \frac{c}{24}} : H_{\mu_1} \otimes \cdots \otimes H_{\mu_n} \longrightarrow \mathbb{C}$$

and denote by

$$\mathcal{H}_\lambda(D; p_1, \dots, p_n; \mu_1, \dots, \mu_n)$$

the vector space of linear operators $\mathrm{Tr}_{H_\lambda} q^{L_0 - \frac{c}{24}}$

for any $\psi \in \mathcal{H}(0, p_1, \cdots, p_n, \infty; \lambda, \mu_1, \cdots, \mu_n, \lambda^*)$

$\longrightarrow$ Define the space of conformal blocks for the torus E by

$$\mathcal{H}(E; p_1, \cdots, p_n; \mu_1, \cdots, \mu_n) = \bigoplus_{0 \le \lambda \le k} \mathcal{H}_\lambda(D; p_1, \cdots, p_n; \mu_1, \cdots, \mu_n)$$

$\longrightarrow$ for <u>$n=0$:</u> basis of $\mathcal{H}(E)$ is given by

$$\chi_\lambda(\tau) = \mathrm{Tr}_{H_\lambda} q^{L_0 - \frac{c}{24}}, \quad \lambda = 0, 1, \cdots, k$$

"characters of affine Lie algebra $A_1^{(1)}$".

We have:

$$\chi_\lambda\left(-\frac{1}{\tau}\right) = \sum_\mu S_{\lambda\mu} \chi_\mu(\tau),$$

$$\chi_\lambda(\tau+1) = \exp 2\pi\sqrt{-1}\left(\Delta_\lambda - \frac{c}{24}\right) \chi_\lambda(\tau)$$

where $S_{\lambda\mu}$ and Δ_λ are given by

$$S_{\lambda\mu} = \sqrt{\frac{2}{k+2}} \; \frac{\sin(\lambda+1)(\mu+1)}{k+2}$$

$$\Delta_\lambda = \frac{\lambda(\lambda+2)}{4(k+2)}$$

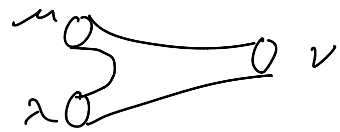
Put $S = (S_{\lambda\mu})$ and $\mathrm{Diag}\left(\exp 2\pi\sqrt{-1}\left(\Delta_\lambda - \frac{c}{24}\right)\right)$, $0 \le \lambda \le k$.

$\longrightarrow S^2 = (ST)^3 = I$.

Let R_k be complex vector space with basis v_λ, $0 \le \lambda \le k$.

Define $v_\lambda \cdot v_k = \sum_\nu N_{\lambda\mu}^{\nu} v_\nu$ with linear extension on R_k. Here

$$N_{\lambda\mu}^{\nu} = \dim \mathcal{H}(p_1, p_2, p_3 ; \lambda, \mu, \nu^*)$$



For $\mathfrak{g} = sl_2(\mathbb{C})$: $N_{\lambda\mu}^{\nu} = N_{\lambda\mu\nu}$ and $N_{\lambda\mu}^{\nu} = 0$ or 1.

<u>Proposition 5:</u>

The algebra R_k is commutative and associative.

<u>Proof:</u>

The commutativity follows from $N_{\lambda\mu}^{\nu} = N_{\mu\lambda}^{\nu}$

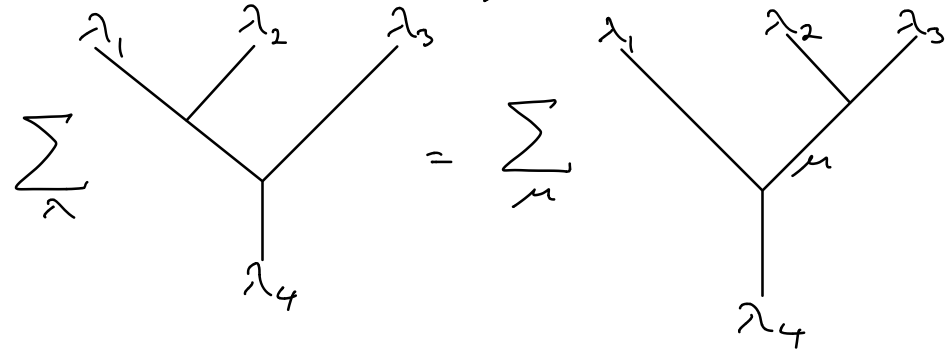
Associativity:

$$(v_{\lambda_1} \cdot v_{\lambda_2}) \cdot v_{\lambda_3} = \sum_{\lambda,\lambda_4} N_{\lambda_1\lambda_2}^{\lambda} N_{\lambda\lambda_3}^{\lambda_4} v_{\lambda_4}$$

$$v_{\lambda_1} \cdot (v_{\lambda_2} \cdot v_{\lambda_3}) = \sum_{\mu,\lambda_4} N_{\lambda_1\mu}^{\lambda_4} N_{\lambda_2\lambda_3}^{\mu} v_{\lambda_4}$$

equality follows from:

$$\sum_{\lambda} N^{\lambda}_{\lambda_1\lambda_2}\, N^{\lambda_4}_{\lambda\lambda_3} = \sum_{\mu} N^{\lambda_4}_{\lambda_1\mu}\, N^{\mu}_{\lambda_2\lambda_3}$$



$\square$

The algebra R_k is called "Verlinde algebra" or "fusion algebra" for the $SU(2)$ Wess-Zumino-Witten model at level k.

It can be shown that

$$\phi : \mathbb{C}[X]/(X^{k+1}) \longrightarrow R_k \quad (*)$$

defined by $\phi(X) = v_1$ is isomorphism.

<u>Proposition 6:</u>

$$\begin{aligned} N_{\lambda\mu\nu} &= \dim H(p_1, p_2, p_3; \lambda, \mu, \nu) \\ &= \sum_{\alpha} \frac{S_{\lambda\alpha} S_{\mu\alpha} S_{\nu\alpha}}{S_{0\alpha}} \end{aligned}$$

"Verlinde formula"

Proof:

Denote $(N_\lambda) = N_{\lambda\mu\nu}$, $0 \le \mu, \nu \le k$

$(k+1) \times (k+1)$ matrix. For $\lambda = 1$: $N_{1\mu\nu} = 1$ if $|\mu - \nu| = 1$ and $N_{1\mu\nu} = 0$ else.

Check that matrix N_1 is diagonalized by matrix S with eigenvalue $S_{\lambda 1}/S_{\lambda 0}$, $\lambda = 0, 1, \dots, k$. ϑ_λ, $\lambda \ge 1$ is polynomial in ϑ_1 (see (*))

$\longrightarrow$ N_λ, $\lambda \ge 1$ is polynomial in N_1.

$\Rightarrow$ N_λ is diagonalized by S as well, with eigenvalues $S_{\lambda\mu}/S_{0\mu}$, $\mu = 0, 1, \dots, k$ (exercise).

$\longrightarrow$ Verlinde formula

$\square$

Next: Basis of conformal blocks on sphere

$\mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n)$

Take μ_j, $0 \le j \le n$ level k highest weights and $\mu_0 = \mu_n = 0$ s.t. $(\mu_{j-1}, \lambda_j, \mu_j)$ satisfies quantum Clebsch-Gordan rule at level k. Consider chiral vertex operators

$$\Psi_j(z) : H_{\mu_{j-1}} \otimes H_{\lambda_j} \longrightarrow \overline{H}_{\mu_j}, \quad 1 \le j \le n,$$

and associated operators

$$\phi_j(z, \xi_j) : H_{\mu_{j-1}} \longrightarrow \overline{H}_{\mu_j}, \quad \xi_j \in H_{\lambda_j}, \quad 1 \le j \le n$$

Then the composition

$$\phi_n(z_n, \xi_n) \cdots \phi_1(z_1, \xi_1) : H_0 \longrightarrow \overline{H}_0$$

together with the correspondence

$$\xi_1 \otimes \cdots \otimes \xi_n \longrightarrow \langle v_0^*, \phi_n(z_n, \xi_n) \cdots \phi_1(z_1, \xi_1) v_0 \rangle$$

$$(v_0 \in H_0, \; v_0^* \in H_0^*)$$

determines a multilinear map

$$\Psi_{\mu_0 \mu_1 \cdots \mu_n}(z_1, \cdots, z_n) : H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n} \to \mathbb{C}$$

$\longrightarrow$ restriction on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ satisfies KZ equation.

Proposition 7:

$$\dim \mathcal{H}(p_1, \cdots, p_n ; \lambda_1, \cdots, \lambda_n)$$

$$= \sum_{0 \le \lambda \le k} \frac{S_{\lambda_1 \lambda} \cdots S_{\lambda_n \lambda}}{(S_{0\lambda})^{n-2}}$$

Proof:

Suppose $n \geq 3$. Then

$$\dim \mathcal{H}(p_1, \ldots, p_n; \lambda_1, \ldots, \lambda_n)$$

$$= \sum_{\mu_1, \ldots, \mu_{n-1}} \prod_{j=1}^{n} N_{\mu_{j-1} \lambda_j \mu_j}$$

$$\xrightarrow{\text{Prop. 6}} \sum_{0 \leq \lambda \leq K} \frac{S_{\lambda_1 \lambda} \cdots S_{\lambda_n \lambda}}{(S_{0\lambda})^{n-2}}$$

$\square$

"pairs of pants decomposition."